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Quasi-exactly solvable quasinormal modes

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Abstract
We consider quasinormal modes with complex energies from the point of view of the theory of quasi-exactly solvable (QES) models. We demonstrate that it is possible to find new potentials which admit exactly solvable or QES quasinormal modes by suitable complexification of parameters defining the QES potentials. Particularly, we obtain one QES and four exactly solvable potentials out of the five one-dimensional QES systems based on the $sl(2)$ algebra.

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1. Introduction

Quasinormal modes (QNM) arise as perturbations of stellar or black hole spacetimes [1]. They are solutions of the perturbation equations that are outgoing to spatial infinity and the event horizon. This 'outgoing wave boundary condition' was first adopted by Gamow in his explanation of the α-decay of atoms as a quantum tunnelling process [2]. Generally, these conditions lead to a set of discrete complex eigenfrequencies, with the real part representing the actual frequency of oscillation and the imaginary part representing the damping. QNM carry information of black holes and neutron stars, and thus are of importance to gravitational-wave astronomy. In fact, these oscillations, produced mainly during the formation phase of the compact stellar objects, can be strong enough to be detected by several large gravitational wave detectors under construction. Recently, QNM of particles with different spins in black hole spacetimes have also received much attention [3].

Owing to the intrinsic complexity in solving the perturbation equations in general relativity with the appropriate boundary conditions, one has to resort to various approximation methods, e.g., the WKB method, the phase-integral method etc, in obtaining QNM solutions. It is therefore helpful that one can get some insights from exact solutions in simple models, such as the inverted harmonic oscillator [4, 5] and the Pöschl–Teller potential [6]. Unfortunately, the number of exactly solvable models is rather limited.

Recently, in non-relativistic quantum mechanics a new class of potentials which are intermediate to exactly solvable ones and non-solvable ones has been found. These are called quasi-exactly solvable (QES) problems for which it is possible to determine analytically a part of the spectrum but not the whole spectrum [7–12]. The discovery of this class of spectral problems has greatly enlarged the number of physical systems which we can study analytically. In the last few years, QES theory has also been extended to the Pauli and Dirac equations [13].

In this paper we would like to study solutions of QNM based on the Lie-algebraic approach of QES theory. We demonstrate that, by suitable complexification of some parameters of the generators of the $sl(2)$ algebra while keeping the Hamiltonian Hermitian, we can indeed obtain potentials admitting exact or quasi-exact QNMs. Such consideration has not been attempted before in studies of QES theory. Our work represents a direct opposite of the work in [14], where QES *real energies* were obtained from a *non-Hermitian* $\mathcal{PT}$-symmetric Hamiltonian.

2. QES theory

Let us briefly review the essence of the Lie-algebraic approach [7–9] to QES models[1]. Consider a Schrödinger equation $H\psi = E\psi$ with Hamiltonian $H = -d_x^2 + V(x)$ ($d_x \equiv \mathrm{d}/\mathrm{d}x$) and wavefunction $\psi(x)$. Here x belongs either to the interval $(-\infty, \infty)$ or $[0, \infty)$. Now suppose we make an 'imaginary gauge transformation' on the function ψ: $\psi(x) = \chi(x)\,\mathrm{e}^{-g(x)}$, where $g(x)$ is called the gauge function. For physical systems which we are interested in, the phase factor $\exp(-g(x))$ is responsible for the asymptotic behaviour of the wavefunction so as to ensure normalizability. The function $\chi(x)$ satisfies a Schrödinger equation with a gauge transformed Hamiltonian $H_g = \mathrm{e}^{g} H\,\mathrm{e}^{-g}$. Suppose H_g can be written as a quadratic combination of the generators J^a of some Lie algebra with a finite dimensional representation. Within this finite dimensional Hilbert space the Hamiltonian H_g can be diagonalized, and therefore a finite number of eigenstates are solvable. Then the system described by H is QES. For one-dimensional QES systems the most general Lie algebra is $sl(2)$, and H_g can be expressed as

$$H_g = \sum C_{ab} J^a J^b + \sum C_a J^a + \text{real constant}, \tag{1}$$

where C_{ab}, C_a are taken to be *real constants* in [8, 9]. The generators J^a of the $sl(2)$ Lie algebra take the differential forms: $J^+ = z^2 d_z - nz$, $J^0 = z d_z - n/2$, $J^- = d_z$ ($n = 0, 1, 2, \ldots$). The variables x and z are related by some function to be described later. n is the degree of the eigenfunctions χ, which are polynomials in a $(n+1)$-dimensional Hilbert space with the basis $\langle 1, z, z^2, \ldots, z^n\rangle$.

Substituting the differential forms of J^a into equation (1), one sees that every QES operator H_g can be written in the canonical form: $H_g = -P_4(z)d_z^2 + P_3(z)d_z + P_2(z)$, where $P_k(z)$ are the kth degree polynomial in z with real coefficients related to the constants C_{ab} and C_a. The relation between H_g and the standard Schrödinger operator H fixes the required form of the gauge function g and the transformation between the variables x and z. Particularly, $x = \int^z \mathrm{d}y/\sqrt{P_4(y)}$. Analysis on the inequivalent forms of real quartic polynomials P_4 thus gives a classification of all $sl(2)$-based QES Hamiltonians [8, 9]. If one imposes the requirement of non-periodic potentials, then there are only five inequivalent classes, which are called cases 1 to 5 in [9].

Our main observation is this. If some of the coefficients in $P_k(z)$ are allowed to be complex while keeping $V(x)$ real, then all the five cases classified in [9] can indeed support QES/exact quasinormal modes. We shall discuss these cases below.

[1] See [10] for the analytic approach, [11] on classification of one-dimensional QES operators possessing finite-dimensional invariant subspace with a basis of monomials, and [12] on formulation extending to nonlinear operators.

3. QES QNM

We consider case 3 in [9], which corresponds to class I potential in Turbiner's scheme [8]. There are two subclasses in this case, namely, case (3a) and case (3b). We shall present the analysis of QNM potential for case (3a). The other case turns out to give the same potential with a suitable choice of the parameters. The potential in case (3a) has the form (in this paper we adopt the notation of [9])

$$V(x) = A\,\mathrm{e}^{2\sqrt{\nu}x} + B\,\mathrm{e}^{\sqrt{\nu}x} + C\,\mathrm{e}^{-\sqrt{\nu}x} + D\,\mathrm{e}^{-2\sqrt{\nu}x}, \tag{2}$$

where $x \in (-\infty, \infty)$ and ν is a positive scale factor. Note that $V(x)$ is defined up to a real constant, which we omit for simplicity, as it merely shifts the real part of the energy. This remark also applies to the other cases. $V(x)$ in equation (2) reduces to the exactly solvable Morse potentials when $A = B = 0$, or $C = D = 0$. This potential is QES when the coefficients are related by

$$\begin{aligned} &A = \frac{\hat{b}^2}{4\nu}, \qquad B = \frac{\hat{c} + (n+1)\nu}{2\nu}\hat{b},\\ &C = \frac{\hat{c} - (n+1)\nu}{2\nu}\hat{d}, \qquad D = \frac{\hat{d}^2}{4\nu},\\ &n = 0, 1, 2 \ldots. \end{aligned} \tag{3}$$

Here $\hat{b}$, $\hat{c}$, $\hat{d}$ are arbitrary real constants. For each integer $n \geqslant 0$, there are $n+1$ exactly solvable eigenfunctions in the $(n+1)$-dimensional QES subspace:

$$\psi_n(x) = \exp\left[\frac{\hat{b}}{2\nu}\,\mathrm{e}^{\sqrt{\nu}x} + \frac{\hat{c} - n\nu}{2\sqrt{\nu}}x - \frac{\hat{d}}{2\nu}\,\mathrm{e}^{-\sqrt{\nu}x}\right]\chi_n(\mathrm{e}^{\sqrt{\nu}x}). \tag{4}$$

Here $\chi_n(z)$ is a polynomial of degree n in $z = \exp(\sqrt{\nu}x)$. To guarantee the normalizability of the eigenfunctions, the real constants $\hat{b}$, $\hat{c}$, $\hat{d}$, ν and n must satisfy certain relations [9].

We want to see if we can get QNM solutions if we allow some parameters to be complex, while still keeping the potential $V(x)$ real. This latter requirement restricts the possible values of the parameters, and hence the forms of QES potential admitting quasinormal modes. For the case at hand, we find that one possible choice of values of $\hat{b}$, $\hat{c}$ and $\hat{d}$ is

$$\hat{b} = \mathrm{i}b, \qquad \hat{c} = -(n+1)\nu, \qquad \hat{d} = d, \qquad b, d: \text{ real constants.} \tag{5}$$

The potential equation (2) becomes

$$V_n(x) = -\frac{b^2}{4\nu}\,\mathrm{e}^{2\sqrt{\nu}x} - (n+1)\,d\,\mathrm{e}^{-\sqrt{\nu}x} + \frac{d^2}{4\nu}\,\mathrm{e}^{-2\sqrt{\nu}x}, \tag{6}$$

and the wavefunction equation (4) becomes

$$\psi_n(x) = \exp\left[\frac{\mathrm{i}b}{2\nu}\,\mathrm{e}^{\sqrt{\nu}x} - \left(n + \frac{1}{2}\right)\sqrt{\nu}x - \frac{d}{2\nu}\,\mathrm{e}^{-\sqrt{\nu}x}\right]\chi_n(\mathrm{e}^{\sqrt{\nu}x}). \tag{7}$$

$V(x)$ approaches $\mp\infty$ as $x \to \pm\infty$, respectively: it is unbounded from below on the right. For small positive d and sufficiently large n, $V(x)$ can have a local minimum and a local maximum. In this case, the well gets shallower as d increases at fixed value of n, or as n decreases at fixed d. Figure 1 presents a schematic sketch of $V(x)$ with $b = 1$ and $n = 1$. We emphasize here that for the different value of n, each $V(x)$ represents a different QES potential admitting $n+1$ QES solutions. Since $V(x) \to \infty$ as $x \to -\infty$, the wavefunction must vanish in this limit. This means $d > 0$ from equation (7). For the outgoing boundary

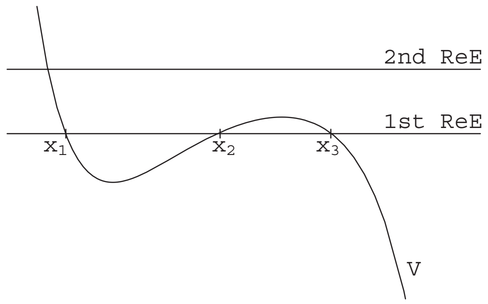

Figure 1. Schematic sketch of the potential $V(x)$ in equation (6) and the real parts of the corresponding QNMs with $b = 1$ and $n = 1$.

condition, we must take $b > 0$. Before we go on, we note here that there is another possible choice of the parameters, namely,

$$\hat{b} = -b, \qquad \hat{c} = (n+1)\nu, \qquad \hat{d} = \mathrm{i}d, \qquad b, d: \text{ real constants.} \tag{8}$$

However, this choice leads to a potential related to equation (6) by the reflection $x \mapsto -x$. Hence, we will only discuss the potential in equation (6) here.

To see that the wavefunctions $\psi_n(x)$ do represent quasinormal modes, we determine the corresponding energy E_n. This is easily done by solving the eigenvalue problem of the polynomial part $\chi_n(z)$ of the wavefunction. From the Schrödinger equation we find that for $n = 0$, the energy is $E_0 = -\nu/4 - \mathrm{i}bd/2\nu$. This clearly shows that the only QES solution when $n = 0$ is a QNM with an energy having a negative imaginary part (recall that $b, d, \nu > 0$). For $n = 1$, we have two QES solutions. Their energies are $E_1 = -5\nu/4 - \mathrm{i}bd/2\nu \pm \sqrt{\nu^2 - \mathrm{i}bd}$. Again we have two QNM modes. One can proceed accordingly to obtain $n + 1$ QNM modes with higher values of n. However, for large n, computation becomes tedious, and one has to resort to numerical means. For definiteness, we list in table 1 some values of E_n for the case where $b = d = \nu = 1$. In this case, the potential has a local minimum and a local maximum, with the barrier height rises as n increases. For large n, some states (in parentheses) have their real parts of energy lie below the local maximum of the barrier. Such states are conventionally called the metastable states. We see from the table that the number of metastable states contained within the well increases as n becomes larger. This is reasonable as the well becomes deeper as n increases.

We have also used the WKB method to determine possible metastable states that can be trapped by the well. The method is as follows. Let the three turning points, from the left to the right, be designated as x_1, x_2 and x_3, as shown in figure 1. The boundary conditions of QNM then leads to the following quantization condition for the energy E:

$$\mathrm{e}^{2\mathrm{i}\beta} = \frac{1 + 4\,\mathrm{e}^{2\gamma}}{1 - 4\,\mathrm{e}^{2\gamma}}, \tag{9}$$

where $\beta \equiv \int_{x_1}^{x_2} \mathrm{d}x \sqrt{E - V(x)}$ and $\gamma \equiv \int_{x_2}^{x_3} \mathrm{d}x \sqrt{V(x) - E}$.

To the first approximation, we set $\mathrm{e}^{-2\gamma} \approx 0$ and $E \approx \mathrm{Re}(E)$. This gives the quantization condition of $\mathrm{Re}(E)$: $\beta = (l + 1/2)\pi, l = 0, 1, 2\ldots.$ Then by keeping the imaginary part of E in β, but dropping it in γ, we can obtain an estimate of $\mathrm{Im}(E)$:

$$\mathrm{Im}(E) = -\frac{\exp\left(-2\int_{x_2}^{x_3} \mathrm{d}x \sqrt{V(x) - \mathrm{Re}(E)}\right)}{2\int_{x_1}^{x_2} \mathrm{d}x / \sqrt{\mathrm{Re}(E) - V(x)}}. \tag{10}$$

Table 1. Values of QNM energy E_n for the QES potentials in case 3 with parameters $b = d = \nu = 1$. Note that for each n, there are $n + 1$ values of E_n. Energy levels of metastable states are in parentheses. WKB estimates of energy of metastable states are also listed.

n	E_n (QES)	E_n (WKB)
0	$-0.25 - 0.5\text{i}$	–
1	$(-2.349 - 0.0449\text{i})$	$-2.313 - 0.0588\text{i}$
	$-0.151 - 0.955\text{i}$	–
2	$(-6.271 - 0.000\,140\text{i})$	$-6.267 - 0.000\,147\text{i}$
	$(-2.447 - 0.135\text{i})$	$-2.439 - 0.156\text{i}$
	$-0.0317 - 1.365\text{i}$	–
3	$(-12.258 - 4.786 \times 10^{-8}\text{i})$	$-12.257 - 4.813 \times 10^{-8}\text{i}$
	$(-6.293 - 0.000\,817\text{i})$	$-6.284 - 0.000\,902\text{i}$
	$-2.542 - 0.259\text{i}$	–
	$0.0939 - 1.740\text{i}$	–
4	$(-20.255 - 4.310 \times 10^{-12}\text{i})$	$-20.254 - 4.328 \times 10^{-12}\text{i}$
	$(-12.265 - 3.810 \times 10^{-7}\text{i})$	$-12.263 - 4.010 \times 10^{-7}\text{i}$
	$(-6.323 - 0.002\,742\text{i})$	$-6.306 - 0.003\,11\text{i}$
	$-2.628 - 0.406\text{i}$	–
	$0.220 - 2.091\text{i}$	–

In table 1 we have also listed the WKB results for the metastable states. It is interesting to note that all the metastable states obtained by WKB methods are in fact the QES states in the cases we considered. The exact values of the QES energies and those of WKB calculations are seen to be consistent.

4. Oscillator-like potentials

We now turn to the other four cases. These four cases admit *exact* QNM solutions. Here we shall discuss cases 4 and 5 of [9], which are associated with oscillator potentials.

For case 4, the potential is given by

$$V(x) = Ax^6 + Bx^4 + Cx^2 + \frac{D}{x^2}, \qquad x \in [0, \infty). \tag{11}$$

It is QES if the coefficients are related by

$$\begin{aligned} A &= \frac{\hat{b}^2}{256}, \qquad B = \frac{\hat{b}\hat{c}}{32}, \\ C &= \frac{1}{16}[\hat{c}^2 + (2\hat{d} + 3(n+1))\hat{b}], \\ D &= \left(\hat{d} - \frac{n}{2}\right)\left(\hat{d} - \frac{n}{2} - 1\right), \qquad n = 0, 1, \ldots, \end{aligned} \tag{12}$$

with real constants $\hat{b}$, $\hat{c}$ and $\hat{d}$. As before, we now relax the reality constraint on the parameters but keeping $V(x)$ real, and determine if QNMs can be supported in this case. It turns out that the answer is positive if we let $\hat{b} = 0$, $\hat{c} = 4\text{i}a$, and $\hat{d} = d$, with real a and d. This leads to the potential

$$V(x) = -a^2x^2 + \frac{\gamma(\gamma - 1)}{x^2}. \tag{13}$$

Here $\gamma \equiv d - \frac{n}{2}$ is arbitrary. Classes VII and VIII in [8] also lead to this potential when some parameters are allowed to be complex. Equation (13) is independent of n, and hence one can solve for solvable states with any degree n in its polynomial part. This system is therefore exactly solvable. For $\gamma > 1$ and $\gamma < 0$, $V(x)$ is a monotonic decreasing function on the positive half-line. If $0 < \gamma < 1$, $V(x) \to -\infty$ at $x = 0, \infty$, and has a global maximum in between.

The wavefunctions take the form $\psi_n = x^\gamma\, \mathrm{e}^{\mathrm{i}ax^2/2}\chi_n(x^2)$, where $\chi_n(x^2)$ is an nth degree polynomial in $z \equiv x^2$. In order that the wavefunction satisfies the outgoing wave condition and vanishes at the origin, we must have $a > 0, \gamma > 0$. Inserting $\psi_n(x)$ into the Schrödinger equation, we determine the energies to be $E_n = -\mathrm{i}(4n+2\gamma+1)a$. Since a and γ are positive, E_n is always negative, indicating decaying QNMs.

Cases 3 and 4 illustrate the main steps in complexifying the relevant parameters to obtain QES/exact QNMs. It is satisfying to find that the remaining three cases can also be extended to give potentials which also admit exact QNMs. These cases will be treated briefly below.

For case 5, we find that the only viable choice of potential is $V(x) = -(cx+d)^2 + d^2/4$, where c and d are real constants. This is a shifted inverted oscillator. Just as in case 4, here $V(x)$ is also independent of n, and hence also exactly solvable. We mention that class VI in [8] also leads to this potential with an appropriate choice of parameters.

For simplicity, we briefly discuss the case with $d = 0$: $V(x) = -c^2x^2/4$ [4]. QNM solutions in such inverted oscillator have been discussed in [5] using the modified annihilation and creation operators. Here we consider the problem from the point of QES potential. The wavefunction is given by $\psi_n(x) = \exp(\mathrm{i}cx^2/4)\chi_n(x)$. From the Schrödinger equation, we get the energies $E_n = -\mathrm{i}c(n+1/2)$. For $c > 0$, the wavefunction describes decaying outgoing QNM away from the maximum $x = 0$ to $x = \pm\infty$. This is analogous to the QNM in black holes. When $c < 0$, we have growing incoming QNM moving towards the origin. This latter case was obtained in [5].

5. Hyperbolic potentials

Potentials in cases 1 and 2 involve hyperbolic functions. A proper complexification of the parameters in case 1 is

$$V(x) = -\frac{cd}{2\nu}\tanh(\sqrt{\nu}x)\,\mathrm{sech}(\sqrt{\nu}x) + \frac{1}{4\nu}(\nu^2 + c^2 - d^2)\,\mathrm{sech}^2(\sqrt{\nu}x), \qquad (14)$$

where $x \in (-\infty, \infty)$. Its normalizable counterpart is, according to the classification in [15], the exactly solvable Scarf II potential. The special case where $d = 0$, which is the inverted Pöschl–Teller potential, has been employed in [6] in their study of black holes' QNMs. We can easily obtain the wavefunctions and energies for the general case, equation (14), from QES theory. The wavefunction has the form: $\psi_n(x) = (\cosh\sqrt{\nu}x)^{(\mathrm{i}c+\nu)/2\nu}\exp(\mathrm{i}d\tan^{-1}(\sinh\sqrt{\nu}x)/2\nu)\chi_n(\sinh\sqrt{\nu}x)$, and the corresponding energy is

$$E_n = \frac{c^2}{4\nu} - \left(n + \frac{1}{2}\right)^2\nu - \mathrm{i}c\left(n + \frac{1}{2}\right). \qquad (15)$$

Note that E_n is independent of d, which is a general feature of Scarf-type potentials. As in case 5, the imaginary part is proportional to $n + 1/2$, which is the characteristic of black hole QNMs. We mention here that there is another complexification scheme for case 1, giving a singular potential which exhibits very peculiar features, such as the existence of continuous bound states spectrum. As this is not related to QNM, it will be reported elsewhere [16].

Case 2 potential with QNM is

$$V(x) = -\frac{cd}{2\nu}\coth(\sqrt{\nu}x)\,\mathrm{cosech}\,(\sqrt{\nu}x) - \frac{1}{4\nu}(\nu^2 + c^2 + d^2)\,\mathrm{cosech}^2(\sqrt{\nu}x), \qquad (16)$$

where $x \in (0, \infty)$. Its normalizable counterpart is the generalized Pöschl–Teller potential [15]. The wavefunction is $\psi_n(x) = (\sinh\sqrt{\nu}x)^{(ic+\nu)/2\nu}(\tanh(\sqrt{\nu}x/2))^{id/2\nu}\chi_n(\cosh\sqrt{\nu}x)$. The energies are exactly given by equation (15). The two cases share the same QNM spectrum, despite the difference in the forms of the potentials and wavefunctions. This is not surprising, as it only reflects the same relation between the two counterpart potentials, the Scarf II and the generalized Pöschl–Teller potential (see e.g., table 1 in [15]).

To summarize, we have demonstrated that it is possible to extend the usual QES theory to accommodate QNM solutions, by complexifying certain parameters defining the QES potentials. We found that the five $sl(2)$-based QES systems listed in [9] can be so extended. While one of these cases admits QES QNM, the other four cases give exact QNM solutions. It is hoped that our work will motivate the search of many more exact/quasi-exact systems of QNM in QES theories based on higher Lie algebras, and in higher dimensions.

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